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The Physical Meaning of Gauge Transformations in Classical Electrodynamics

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Abstract:

The structure of electrodynamics based on the variational principle together with causality and space-time homogeneity is analyzed. It is proved that in this case the 4-potential is defined uniquely. Therefore, the approach where Maxwell equations and the Lorentz law of force are regarded as cornerstones of the theory is *not equivalent* to the one described above.

One may regard the equations of motion of a physical system as the fundamental elements of a theory. Thus, the equations of motion can be used for deriving useful formulas that describe properties of the system. However, it is now acceptable that other principles take a more profound role. Using this approach, the variational principle, causality and homogeneity of space-time are regarded here as the basis of the discussion. In particular, the fundamental equations of motion of classical electrodynamics, namely, Maxwell equations and the Lorentz law of force can be derived from the variational principle [1,2]. The discussion carried out here proves that in the case of electrodynamics, the two approaches are *not equivalent* and that the variational principle imposes further restrictions on the theory's structure.

It is proved in this work that if one adheres to the variational principle together with causality and space-time homogeneity then the 4-potential of electrodynamics is defined uniquely. Therefore, in this approach, gauge transformations are no more than useful mathematical tricks applied in a process of solving specific problems.

The Lagrangian density used for a derivation of Maxwell equations is (see [1], pp. 73-74; [2], pp. 596-597)

$$\mathcal{L} = -\frac{1}{16\pi}F^{\mu\nu}F_{\mu\nu} - j^{\mu}A_{\mu} \tag{1}$$

In the present work, units where the speed of light $c = 1$ and $\hbar = 1$ are used. Thus, one kind of dimension exists and the length $[L]$ is used for this purpose. Greek indices run from 0 to 3. The metric is diagonal and its entries are (1,-1,-1,-1). The symbol $_{,\mu}$ denotes the partial differentiation with respect to x^{μ}. A_{μ} denotes the 4-potential and $F^{\mu\nu}$ denotes the antisymmetric tensor of the electromagnetic fields

$$F^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} (A_{\beta,\alpha} - A_{\alpha,\beta}) = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}. \tag{2}$$

For the simplicity of the discussion, let us examine the fields associated with one charged particle e whose motion is given. This approach can be justified because, due to the linearity of Maxwell equations, one finds that the fields of a system of charges is a superposition of the fields of each individual charge belonging to the system. Let us examine the electromagnetic fields at a given space-time point x^μ. Using Maxwell equation and the principle of causality, one can derive the retarded Lienard-Weichert 4-potential (see [1], pp. 160-161; [2], pp. 654-656)

$$A_\mu = e \frac{v_\mu}{R^\alpha v_\alpha}. \tag{3}$$

Here v_μ is the charge's 4-velocity at the retarded time and R^μ is the 4-vector from the retarded space-time point to the field point x^μ. This 4-potential defines the fields uniquely.

A gauge transformation of (3) is (see [1], pp. 49-50; [2], pp. 220-223)

$$A'_\mu = A_\mu + \Phi_{,\mu}. \tag{4}$$

In the following lines, the form of the gauge function $\Phi(x^\mu)$ is investigated.

Relying on the variational principle, one finds constraints on terms of the Lagrangian density. An examination of the Lagrangian density (1), proves that every term of this expression is a Lorentz scalar having the dimensions $[L^{-4}]$. Thus, the action is a Lorentz scalar and is dimensionless in the unit system used here. In particular, the 4-potential A_μ must be a 4-vector whose dimension is $[L^{-1}]$. This requirement is satisfied by the Lienard-Weichert 4-potential (3). Thus, also $\Phi_{,\mu}$ of (4) is a 4-vector and Φ must be a dimensionless Lorentz scalar function of space-time coordinates.

Now, the coordinates are entries of a 4-vector. Therefore, the general form of a homogeneous function which is a Lorentz scalar depending on the coordinate must be a sum of power functions of the form

$$f_{a,p}(x^\mu) = [(x^\mu - x_a^\mu)(x_\mu - x_{a\mu})]^p. \tag{5}$$

Here $2p$ denotes the order of the homogeneous function and x_a^μ denotes a specific space-time point. Relying on the homogeneity of space-time, one finds that in the case discussed here there is just one specific point x_a^μ, which is the retarded position of the charge. Thus, in order to be a dimensionless Lorentz scalar, Φ must take the form

$$\Phi(x^\mu) = \frac{c_1[(x^\mu - x_a^\mu)(x_\mu - x_{a\mu})]^p}{c_2[(x^\mu - x_a^\mu)(x_\mu - x_{a\mu})]^p} = const. \tag{6}$$

Here the factors, c_i denote numerical constants.

These arguments complete the proof showing that the gauge function Φ is a constant and the gauge 4-vector $\Phi_{,\mu}$ vanishes identically. Hence, the Lienard-Weichert 4-vector (3) is unique.

The foregoing result indicates the difference between an electrodynamic theory where Maxwell equations and the Lorentz law of force are regarded as the theory's cornerstone and a theory based on the variational principle together with causality and space-time homogeneity. Indeed, if Maxwell equations are the theory's cornerstone then it is very well known that one is free to define the gauge function $\Phi(x^\mu)$ of (4) (see [1], pp. 49-50; [2], pp. 220-223). For this reason, the result of this work proves that the two approaches are *not equivalent.*

References:

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